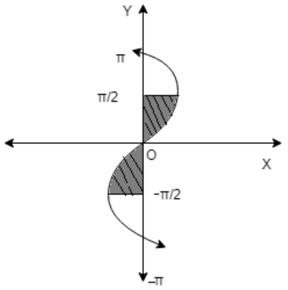
Solution

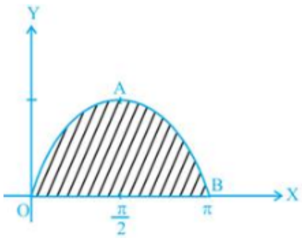
Class 12 - Mathematics

2020-21 paper 5

Part A

1. we have, $\int_0^c 3dx = \frac{16}{3}$
   $3(x)_0^c = \frac{16}{3}$
   $3c = \frac{16}{3}$
   $c = \frac{16}{9}$
2. The required area



   = $2\int_0^{\pi/2} x\ dy$, where y = $\sin^{-1}x$ i.e x = siny
   = $2\int_0^{\pi/2} sin\ y\ dy$
   $(\int sin\ y\ dy = cos\ y + c)$
   =$-2[cos\ y]_0^{\pi/2}$
   =$-2[cos\frac{\pi}{2} - cos\ 0]$
   = 2 sq. units
   The area bounded by the curve y = $sin^{-1}x$ and the lines x = 0, |y|=$\frac{\pi}{2}$ = 2 sq. units
3. 

   We have
   Area $OAB = \int_0^\pi y dx = \int_0^\pi \sin x dx = |-\cos x|_0^\pi$
   $= \cos 0 - \cos\pi = 2$sq units.
4. Required area = the area of the region enclosed by the lines y=x, x=e, and the curve $y = \frac{1}{x}$ and the positive x-axis
   =$\int_0^1 xdx + \int_1^e \frac{1}{x}dx$
   $= \frac{1}{2} + 1$
   $= \frac{3}{2} sq\ units$
5. A function whose anti derivative is $3x^2 + 4x^3$.
   $\frac{d}{dx}\left(x^3 + x^4\right) = 3x^2 + 4x^3$
   Therefore, an anti derivative of $3x^2 + 4x^3$ is $x^3 + x^4$
6. I = $\int_{-2}^3 \frac{1}{x+7}$ dx
   = $[\log (x + 7)]_{-2}^3$
   = [log (3 + 7)- log(-2+7)]
   = log 10 - log 5

=log $\frac{10}{5}$ [$\left[\because \log a-\log b=\log \frac{a}{b}\right]$
= log 2

7. We have,
$$\int x^{\frac{5}{4}} dx=\frac{x^{\frac{5}{4}+1}}{\frac{5}{4}+1}+c=\frac{x^{\frac{5+4}{4}}+c}{\frac{5+4}{4}}=\frac{4x^{\frac{9}{4}}}{9}+c\ ,$$
where c is constant of integration.
8. Given:
$\int(2-5x)(3+2x)(1-x)dx$
$=\int(6-11x-10x^2)(1-x)dx$
$=\int(10x^3+x^2-17x+6)dx$
$=\frac{10x^4}{4}+\frac{x^3}{3}-\frac{17x^2}{2}+6x+c$
$=\frac{5x^4}{2}+\frac{x^3}{3}-\frac{17x^2}{2}+6x+c$,where c is constant of integration.
9. In the given equation, the highest-order derivative is $\frac{d^3y}{dx^3}$ and its power is 2.Therefore,we have,
order = 3 and degree = 2.
10. Given : y = $5e^{7x}+6e^{-7x}$
Then, we have
$\frac{dy}{dx}=5\cdot\frac{d}{dx}\left(e^{7x}\right)+6\cdot\frac{d}{dx}\left(e^{-7x}\right)$
$=5\cdot e^{7x}\cdot\frac{d}{dx}(7x)+6\cdot e^{-7x}\cdot\frac{d}{dx}(-7x)$
$=35e^{7x}-42e^{-7x}$
$\therefore\frac{d^2y}{dx^2}=35\cdot\frac{d}{dx}\left(e^{7x}\right)-42\cdot\frac{d}{dx}\left(e^{-7x}\right)$
$=35\cdot e^{7}\cdot\frac{d}{dx}(7x)-42\cdot e^{-7x}\cdot\frac{d}{dx}(-7x)$
$=7\times 35\ e^{7x}+7\times 42\ e^{-7x}$
$=49\times 5e^{7x}+49\times 6e^{-7x}$
$=49(5\ e^{7x}+6\ e^{-7x})$
= 49y
Henced proved
11. Order = 2
Degree = 1
12. In this equation, the highest order derivative is (y'').So, its order is 2
It has a term sin y'. so it has not degree.
13. It is given that
$I=\int\frac{2\cos x}{3\sin^2 x}dx$
We can write it as
$I=\frac{2}{3}\int\cot x\ cosecx\ dx$
By integrating w.r.t. x
$I=\frac{2}{3}(-\ cosec\ x)+c$
So we get
$I=-\frac{2}{3}\ cosec\ x+c$.
14. $\int\frac{\sin^2 x}{1+\cos x}dx=\int\frac{1-\cos^2 x}{1+\cos x}dx$
$=\int\frac{(1-\cos x)(1+\cos x)}{(1+\cos x)}dx$
$=\int(1-\cos x)dx$
= x - sin x + C
Thus $\int\frac{\sin^2 x}{1+\cos x}dx$=x - sinx + C
15. Given that y = cx +$2c^2$
Differentiating with respect to x, $\frac{dy}{dx}$ = c From equation (i),

y = $x\left(\frac{dy}{dx}\right)+2\left(\frac{dy}{dx}\right)^2$

$2\left(\frac{dy}{dx}\right)^2-x\left(\frac{dy}{dx}\right)-y=0$

So, y = cx + 2 c^2 is the required solutoin of the given equation.

16. $\frac{d^3y}{dx^3}+\frac{d^2y}{dx^2}+\frac{dy}{dx}+y\sin y=0$

In this differential equation, the order of the highest order derivative is 3 and its power is 1. Therefore ,the order of the differential equation is 3 and its degree is 1

Hence, It is a non-linear differential equation, as the exponent of the dependent variable is not equal to 1 (by expanding y. sin y).

Section I

17. i. (b) $\frac{\mathrm{dv}}{\mathrm{dt}}=9.8-\frac{\mathrm{v}}{10}$
    ii. (c) 33 m/sec
    iii. (c) S(t) = 98 t + 1080 $e^{-t/10}$ - 1080
    iv. (a) 38.6 m/sec
    v. (d) 5.98 sec.
18. i. (a) $\frac{dx}{dt}$ = k(200,000 - x)
    ii. (c) ₹ 1,55,555
    iii. (d) ₹ 180,246
    iv. (a) 200,000 - 150,000$\left(\frac{2}{3}\right)^{\mathrm{t}}$
    v. (b) $\frac{2}{3}$

Section II

19. Let $\mathrm{I}=\int\sin^5 x\mathrm{dx}$, then we have

$I=\int\sin^4 x\cdot\sin x dx$

$=\int\left(\sin^2 x\right)^2\sin x dx$

$=\int\left(1-\cos^2 x\right)^2\sin x dx$

$=\int\left(\cos^4 x-2\cos^2 x+1\right)\sin x dx$

Putting cos x = t

$\Rightarrow$ -sin x dx = dt

$\Rightarrow$ sin x dx = -dt

$\therefore I=-\int\left(t^4-2t^2+1\right)dt$

$=-\int t^4dt+2\int t^2dt-\int dt$

$=\frac{-t^5}{5}+\frac{2t^3}{3}-t+C$

$=\frac{-\cos^5 x}{5}+\frac{2}{3}\cos^3 x-\cos x+C$ [$\because$ t = cos x]

20. Let I = $\int x^2\sqrt{a^6-x^6}$dx, then we have

I = $\int x^2\sqrt{\left(a^3\right)^2-\left(x^3\right)^2}dx$

Putting x^3 = t

$\Rightarrow$ $3x^2$ dx = dt

$\Rightarrow$ x^2 dx = $\frac{dt}{3}$

$\therefore$ I = $\frac{1}{3}\int\sqrt{\left(a^3\right)^2-t^2}dt$

= $\frac{1}{3}\left[\frac{t}{2}\sqrt{\left(a^3\right)^2-t^2}+\frac{\left(a^3\right)^2}{2}\sin^{-1}\left(\frac{t}{a^3}\right)\right]$ + C

= $\frac{x^3}{6}\sqrt{a^6-x^6}+\frac{a^6}{6}\sin^{-1}\left(\frac{x^3}{a^3}\right)$ + C

21. Here, a = 2, b = 4 and f (x) = 2^x. Therefore, $h=\frac{4-2}{n}\Rightarrow nh=2$

Substituting these values in the equation,

$\int_a^b f(x)dx = \lim_{h\to 0}$ h[f(a) + f(a + h) + f(a + 2h) + ... + f(a + (n - 1)h)], we get

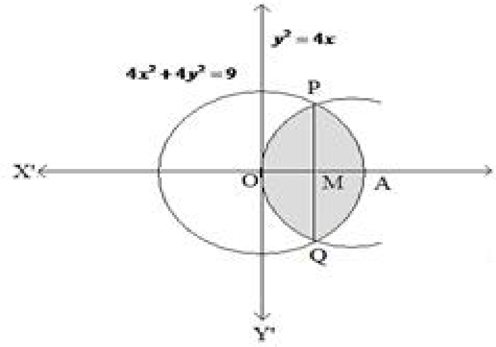
$\int_2^4 2^x dx = \lim_{h\to 0}$ h[f(2) + f(2 + h) + f(2 + 2h) + ... + f(2 + (n - 1)h)]

$\Rightarrow \quad \int_2^4 2^x dx = \lim_{h\to 0} h\left\{2^2 + 2^{2+h} + 2^{2+2h} + \ldots + 2^{2+(n-1)h}\right\}$

$\Rightarrow \quad \int_2^4 2^x dx = \lim_{h\to 0} 4h\left\{1 + 2^h + 2^{2h} + \ldots + 2^{(n-1)h}\right\}$

$\Rightarrow \quad \int_2^4 2^x dx = \lim_{h\to 0} 4h\left\{\frac{(2^h)^n - 1}{2^h - 1}\right\} = 4\lim_{h\to 0}\left\{\frac{2^{nh}-1}{\left(\frac{2^h-1}{h}\right)}\right\}$

$\Rightarrow \quad \int_2^4 2^x dx = 4 \times \left(\frac{2^2-1}{\log 2}\right) = \frac{12}{\log 2}\left[\because nh = 2 \text{ and } \lim_{h\to 0}\frac{2^h-1}{h} = \log 2\right]$

22. 

Points of intersection of the curve are $\left(\frac{1}{2}, \sqrt{2}\right)$ and $\left(\frac{1}{2}, -\sqrt{2}\right)$

The required area is OPAQO and it is symmetrical about x-axis

$\therefore$ Area OPAQO = 2 area OPA

$\text{Area} = 2\left[\int_0^{1/2} 2\sqrt{x}dx + \int_{1/2}^{3/2}\sqrt{\left(\frac{3}{2}\right)^2 - x^2 dx}\right]$

$= 2\left[2\left[\frac{x^{3/2}}{\frac{3}{2}}\right]_0^{1/2}\right] + 2\left[\frac{x}{2}\sqrt{\frac{9}{4} - x^2} + \frac{\frac{9}{4}}{2}\sin^{-1}\left(\frac{x}{\frac{3}{2}}\right)\right]_{1/2}^{3/2}$

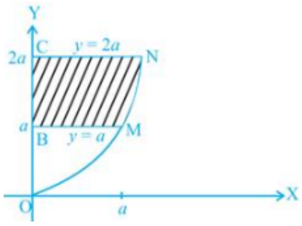
$= \frac{8}{3}\left[\left(\frac{1}{2}\right)^{3/2} - 0\right] + \left[\left(\frac{3}{2}(0) + \frac{9}{4}\sin^{-1}(1)\right) - \left(\frac{1}{2}\sqrt{\frac{9}{4} - \frac{1}{4}} + \frac{9}{4}\sin^{-1}\frac{1}{3}\right)\right]$

$= \frac{8}{3}\left(\frac{1}{2\sqrt{2}}\right) + \frac{9}{4}\left(\frac{\pi}{2}\right) - \frac{1}{\sqrt{2}} - \frac{9}{4}\sin^{-1}\left(\frac{1}{3}\right)$

$= \frac{\sqrt{2}}{6} + \frac{9\pi}{8} - \frac{9}{4}\sin^{-1}\left[\frac{1}{3}\right]$

$= \left(\frac{2\sqrt{2}}{3} - \frac{\sqrt{2}}{2}\right) + \frac{9\pi}{8} - \frac{9}{4}\sin^{-1}\left(\frac{1}{3}\right)$

$= \left[\frac{\sqrt{2}}{6} + \frac{9\pi}{8} - \frac{9}{4}\sin^{-1}\left(\frac{1}{3}\right)\right] = \left(\frac{9\pi}{8} - \frac{9}{4}\sin^{-1}\left(\frac{1}{3}\right) + \frac{1}{3\sqrt{2}}\right) sq.units$

23. 

We have

Area BMNC $\int_0^{2a} xdy = \int_0^{2a} a^{1/3}y^{2/3}dy$

$= \frac{3a^{\frac{1}{3}}}{5}\left[y^{\frac{5}{3}}\right]_a^{2a}$

$= \frac{3a^{\frac{1}{3}}}{5}\left[(2a)^{\frac{5}{3}} - a^{\frac{5}{3}}\right]$

$= \frac{3}{5}a^{\frac{1}{3}}a^{\frac{5}{3}}\left[(2)^{\frac{5}{3}} - 1\right]$

$= \frac{3}{5}a^2\left[2.2^{\frac{2}{3}} - 1\right]$ sq units.

24. We have, y = $ax^3 + bx^2 + c$...(i)
Differentiating both sides of (i) with respect to x, we get
$\frac{dy}{dx} = 3ax^2 + 2bx$ ...(ii)
Differentiating both sides of (ii) with respect to x, we get
$\frac{d^2y}{dx^2} = 6ax + 2b$ ...(iii)
Differentiating both sides of (iii) with respect to x, we get
$\frac{d^3y}{dx^3} = 6a$
Hence, the given function is the solution to the given differential equation.

OR

Let I = $\int \frac{x^3}{(x-1)(x-2)(x-3)} dx$
= $\int 1 + \frac{6x^2-9x+6}{(x-1)(x-2)(x-3)} dx$
Let $\frac{6x^2-11x+6}{(x-1)(x-2)(x-3)}$ = $\frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-3}$
⇒ $6x^2 - 11x + 6$ = A (x - 2)(x - 3) + B (x - 1) (x - 3) + C (x - 2) (x - 1)
Put x = 1
⇒ 1 = 2A ⇒ A = $\frac{1}{2}$
Put x = 2
⇒ 8 = -B ⇒ B = -8
Put x = 3
⇒ 27 = 2C ⇒ C = $\frac{27}{2}$
Thus,
I = $\int dx + \frac{1}{2} \int \frac{dx}{x-1} - 8 \int \frac{dx}{x-2} + \frac{27}{2} \int \frac{dx}{x-3}$ [Put value of A,B & C]
= x + $\frac{1}{2} \log |x-1| - 8 \log |x-2| + \frac{27}{2} \log |x-3| + c$
Hence,
I = x + $\frac{1}{2} \log |x-1| - 8 \log |x-2| + \frac{27}{2} \log |x-3| + c$.

25. We are given that $3e^x \tan y\, dx + (1 - e^x) \sec^2 y\, dy = 0$
⇒ $3e^x \tan y\, dx + (1 - e^x) \sec^2 y\, dy$
⇒ $\frac{3e^x}{1-e^x} dx = -\frac{\sec^2 y}{\tan y} dy$ [separating variables]
⇒ $3 \int \frac{e^x}{1-e^x} dx = - \int \frac{\sec^2 y}{\tan y} dy$ [Integrating both sides]
⇒ $3 \int \frac{e^x}{e^x-1} dx = \int \frac{\sec^2 y}{\tan y} dy$
⇒ 3 log |e^x - 1| = log |tan |tan y| + log C
⇒ $\log \left(\frac{|e^x-1|^3}{|\tan y|} \right)$ = log C
⇒ $\frac{(e^x-1)^3}{\tan y}$ = C
⇒ $(e^x - 1)^3$ = C tan y, which is required solution of the given differential equation.

26. The given differential equation is $\frac{dy}{dx} = \frac{1+y^2}{1+x^2}$
⇒ $(1 + x^2)\, dy = (1 + y^2)\, dx$
⇒ $\frac{1}{1+y^2} dy = \frac{1}{1+x^2} dx$
⇒ $\int \frac{1}{1+y^2} dy = \int \frac{1}{1+x^2} dx$ [Integrating both sides]
⇒ $\tan^{-1} y = \tan^{-1} x + \tan^{-1} C$
⇒ $\tan^{-1} \left(\frac{y-x}{1+xy} \right)$ = $\tan^{-1} C$
⇒ $\frac{y-x}{1+xy}$ = C
⇒ y - x = C (1 + xy), which is the required solution of given differential equation

27. $\int e^x\left(\log x+\frac{1}{x^2}\right)dx$

$=\int e^x\log x dx-\int\frac{e^x}{x^2}dx$

Taking $f_1(x)$ = logx and $f_2(x) = e^x$ in the first integral and keeping the second integral intact,

$\int e^x\log x dx-\int\frac{e^x}{x^2}dx$

$=\log x\int e^x dx-\int\left[\frac{d}{dx}(\log x)\int e^x dx\right]dx-\int\frac{e^x}{x^2}dx$ [INTEGRATION BY PARTS]

$=e^x\log x-\int\frac{e^x}{x}dx-\int\frac{e^x}{x^2}dx$

$=\mathrm{e}^{\mathrm{x}}\log\mathrm{x}-\left[\frac{1}{\mathrm{x}}\int\mathrm{e}^{\mathrm{x}}\mathrm{dx}-\int\left[\frac{\mathrm{d}}{\mathrm{dx}}\left(\frac{1}{\mathrm{x}}\right)\int\mathrm{e}^{\mathrm{x}}\mathrm{dx}\right]\mathrm{dx}\right]-\int\frac{\mathrm{e}^{\mathrm{x}}}{\mathrm{x}^2}\mathrm{dx}$ [INTEGRATION BY PARTS]

$=e^x\log x-\frac{e^x}{x}+\int\frac{e^x}{x^2}dx-\int\frac{e^x}{x^2}dx+c$

$=e^x\left(\log x-\frac{1}{x}\right)+c$, where c is the integrating constant

OR

Slope of tangent at point (x, y) = $-\frac{x}{y}$

$\frac{dy}{dx}=-\frac{x}{y}$

ydy = -xdx

$\int ydt=-\int xdx$

$\frac{y^2}{2}+\frac{x^2}{2}=c_1$

$x^2+y^2=c$ ...(1)

Given, curve is passing through (3,-4), so

$(3)^2+(-4)^2$ = c

9 + 16 = c

c = 25

So, using equation (1) and we get required equation of curve.

$x^2+y^2=25$

$x^2+y^2=25$

28. We have, $\frac{dy}{dx}$ = $(x + y)^2$

Let x + y = v

$\Rightarrow 1+\frac{dy}{dx}=\frac{dv}{dx}$

$\Rightarrow\frac{dy}{dx}=\frac{dv}{dx}-1$

$\therefore\frac{dv}{dx}$ - 1 = v^2

$\Rightarrow\frac{dv}{dx}$ = v^2 + 1

$\Rightarrow\frac{1}{v^2+1}$ dv = dx

Integrating both sides, we get

$\int\frac{1}{v^2+1}dv=\int dx$

$\Rightarrow$ $\tan^{-1}$ v = x + C

$\Rightarrow$ v = tan (x + C)

$\Rightarrow$ x + y = tan (x + C). Hence It is the required solution of given differential equation.

Part B

29. Let 5x + 3 = $A\frac{d}{dx}\left(x^2+4x+10\right)+B$

$\Rightarrow$ 5x + 3 = A(2x + 4) + B

Now, equating the coefficients of x and constant term on both sides, we get,

2A = 5

$\Rightarrow A=\frac{5}{2}$

4A + B = 3

$\Rightarrow$ B = -7

$\Rightarrow 5x+3=\frac{5}{2}(2x+4)-7$

Again, $\int\frac{5x+3}{\sqrt{x^2+4x+10}}dx=\int\frac{\frac{5}{2}(2x+4)-7}{\sqrt{x^2+4x+10}}dx$

$\Rightarrow \frac{5}{2} \int \frac{2x+4}{\sqrt{x^2+4x+10}} dx - 7 \int \frac{1}{\sqrt{x^2+4x+10}} dx$

Now, let us consider, $\int \frac{2x+4}{\sqrt{x^2+4x+10}} dx$

Let x^2 + 4x + 10 = t

⇒ (2x + 4) dx = dt

$\therefore \int \frac{2x+4}{\sqrt{x^2+4x+10}} dx = \int \frac{dt}{\sqrt{t}} = 2\sqrt{t} = 2\sqrt{x^2+4x+10}$ ......(i)

And, Now let us consider, $\int \frac{1}{\sqrt{x^2+4x+10}} dx$

$\Rightarrow \int \frac{1}{\sqrt{x^2+4x+10}} dx = \int \frac{1}{(\sqrt{x^2+4x+4})+\sqrt{6}} dx$

$\Rightarrow \int \frac{1}{(x+2)^2+(\sqrt{6})^2} dx$

$= \log|(x+2)\sqrt{x^2+4x+10}|$ .....(ii)

using eq. (i) and (ii), we get,

$\Rightarrow \int \frac{5x+3}{\sqrt{x^2+4x+10}} dx = \frac{5}{2}[2\sqrt{x^2+4x+10}] - 7\log(x+2)\sqrt{x^2+4x+10}| + C$

$\Rightarrow \int \frac{5x+3}{\sqrt{x^2+4x+10}} dx = 5\sqrt{x^2+4x+10} - 7\log|(x+2)\sqrt{x^2+4x+10}| + C$

30. Let, $I = \int \frac{(x^2+1)(x^2+4)}{(x^2+3)(x^2-5)} dx$

Consider, $\frac{(x^2+1)(x^2+4)}{(x^2+3)(x^2-5)}$

Put $x^2 = y$

Then, $\frac{(x^2+1)(x^2+4)}{(x^2+3)(x^2-5)} = \frac{(y+1)(y+4)}{(y+3)(y-5)}$

$= \frac{y^2+5y+4}{y^2-2y-15}$

$= \frac{(y^2-2y-15)+(7y+19)}{y^2-2y-15}$

$= 1 + \frac{7y+19}{y^2-2y-15}$

$= 1 + \frac{7y+19}{(y+3)(y-5)}$ ...(i)

Using partial fractions,

$\frac{7y+19}{(y+3)(y-5)} = \frac{A}{y+3} + \frac{B}{y-5}$

$\Rightarrow 7y + 19 = A(y-5) + B(y+3)$

Putting y = 5,

$\Rightarrow 35 + 19 = 8B \Rightarrow B = \frac{54}{8} = \frac{27}{4}$

Putting y = -3

$\Rightarrow -21 + 19 = -8A \Rightarrow A = \frac{(-2)}{(-8)} = \frac{1}{4}$

Thus, $\frac{7y+19}{(y+3)(y-3)} = \frac{1}{4} \cdot \frac{1}{(y+3)} + \frac{27}{4} \cdot \frac{1}{(y-5)}$ ...(ii)

From Equations (i) and (ii),

$I = 1 + \frac{7y+19}{(y+3)(y-5)} = 1 + \frac{1}{4} \cdot \frac{1}{(y+3)} + \frac{27}{4} \frac{1}{(y-5)}$

$\therefore I = \frac{(x^2+1)(x^2+4)}{(x^2+3)(x^2-5)} = \frac{1}{4} \cdot \frac{1}{(y+3)} + \frac{27}{4} \cdot \frac{1}{(y-5)} = 1 + \frac{1}{4} \cdot \frac{1}{(x^2+3)} + \frac{27}{4} \frac{1}{(x^2-5)} \left[\because x^2 = y \right]$

Now, $I = \int \left(1 + \frac{1}{4} \cdot \frac{1}{(x^2+3)} + \frac{27}{4} \frac{1}{(x^2-5)} \right) dx$

$= \int dx + \frac{1}{4} \int \frac{dx}{x^2+3} + \frac{27}{4} \int \frac{dx}{x^2-5}$

$= x + \frac{1}{4} \int \frac{dx}{x^2+(\sqrt{3})^2} + \frac{27}{4} \int \frac{dx}{x^2-(\sqrt{5})^2}$

$= x + \frac{1}{4\sqrt{3}} \tan^{-1}\left(\frac{x}{\sqrt{3}}\right) + \frac{27}{4} \frac{1}{2\sqrt{5}} \log\left|\frac{x-\sqrt{5}}{x+\sqrt{5}}\right| + C \left[\because \int \frac{dx}{x^2+a^2} = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + C \text{ and } \int \frac{dx}{x^2-a^2} \right.$

$\left. = \frac{1}{2a} \log\left|\frac{x-a}{x+a}\right| + C \right]$

$= x + \frac{1}{4\sqrt{3}} \tan^{-1}\left(\frac{x}{\sqrt{3}}\right) + \frac{27}{8\sqrt{5}} \log\left|\frac{x-\sqrt{5}}{x+\sqrt{5}}\right| + C$

31. Let $I=\int \frac{x^2+3x-1}{(x+1)^2}$ dx. Then,

$I=\int \frac{x^2+x+2x-1}{(x+1)^2}$dx

$=\int \frac{x(x+1)+2x-1}{(x+1)^2}$ dx

$=\int \frac{x(x+1)}{(x+1)^2}dx+\int \frac{2x-1}{(x+1)^2}dx$

$=\int \frac{x}{x+1}dx+\int \frac{2x+2-2-1}{(x+1)^2}dx$

$=\int \frac{x+1-1}{x+1}dx+\int \frac{2(x+1)-3}{(x+1)^2}dx$

$=\int \frac{x+1}{x+1}dx-\int \frac{1}{x+1}dx+\int \frac{2(x+1)}{(x+1)^2}dx-3\int \frac{1}{(x+1)^2}dx$

$=\int dx-\int \frac{1}{x+1}dx+2\int \frac{1}{x+1}dx-3\int \frac{1}{(x+1)^2}dx$

= x - log |x + 1| + 2 log |x + 1| + $\frac{3}{x+1}$ + c

= x + log |x + 1| + $\frac{3}{x+1}$ + c

OR

To find: Value of $\int(\sin^{-1} x)^2$ dx

Formula used: $\int$ sin x , dx = cos x+c

We have, I =$\int(\sin^{-1} x)^2$ dx ...(i)

Let $\sin^{-1}x$ = t, x = sint,

$\Rightarrow$ cost $=\sqrt{1-x^2}$

$\Rightarrow \frac{1}{\sqrt{1-x^2}}=\frac{dt}{dx}$

$\Rightarrow \sqrt{1-x^2}\,dt=dx$

$\Rightarrow \sqrt{1-(\sin t)^2}\,dt=dx$

$\Rightarrow \sqrt{1-\sin^2 t}\,dt=dx$

$\Rightarrow$ cost dt = dx

Putting this value in equation (i).Hence the given integral becomes,

I=$\int t^2$ cos tdt

$I=t^2\int \cos t dt-\int\left[\frac{d(t^2)}{dt}\int \cos t\,dt\right]dt$

I=t^2 sin t-$\int$[2 tsin t]dt

Again by parts we have,

$I=t^2\sin t-2\left[-t\cos t+\int 1.\cos t\,dt\right]$

I=t^2 sin t+2 t cost-2 sin t+c

$I=\left(\sin^{-1}x\right)^2x+2\left(\sin^{-1}x\right)\sqrt{1-x^2}-2x+c$

32. The given differential equation is,

$\frac{dy}{dx}=\frac{x^2+y^2}{2xy}$

$\Rightarrow \frac{dy}{dx}=-\left(\frac{y}{2x}\right)^{-1}+\left(\frac{y}{2x}\right)$

$\Rightarrow \frac{dy}{dx}=f\left(\frac{y}{x}\right)$

$\Rightarrow$ the given differential equation is a homogenous equation.

The solution of the given differential equation is:

Put y = vx

$\Rightarrow \frac{\mathrm{dy}}{\mathrm{dx}}=\mathrm{v}+\mathrm{x}\frac{\mathrm{dv}}{\mathrm{dx}}$

$\mathrm{v}+\mathrm{x}\frac{\mathrm{dv}}{\mathrm{dx}}=-\left(\frac{\mathrm{vx}}{2\mathrm{x}}\right)^{-1}+\left(\frac{\mathrm{vx}}{3\mathrm{x}}\right)$

$\Rightarrow \mathrm{x}\frac{\mathrm{dv}}{\mathrm{dx}}=-\left(\frac{\mathrm{v}}{2}\right)^{-1}+\left(\frac{\mathrm{v}}{2}\right)$

$\Rightarrow \mathrm{x}\frac{\mathrm{dv}}{\mathrm{dx}}=-\frac{2}{\mathrm{v}}+\left(\frac{\mathrm{v}}{2}\right)=\frac{-4+\mathrm{v}^2}{2\mathrm{v}}$

$\Rightarrow \frac{2v}{v^2-4}dv = \frac{dx}{x}$

Integrating both the sides we get:

$\Rightarrow \int \frac{2v}{v^2-4}dv = \int \frac{dx}{x} + c$

$\Rightarrow \frac{2}{2}\ln\left|v^2 - 4\right|$ = ln|x| + ln|c|

Resubstituting the value of y = vx we get

$\Rightarrow \ln\left|\left(\frac{y}{x}\right)^2 - 4\right|$ = ln|x| + ln|c|

on simplification, we have,

$(y^2 - 4x^2) = cx^3$

33. Let r be the radius and S be the surface area of the balloon at any time t. Therefore,we have,

S = $4\pi r^2$

$\Rightarrow \frac{dS}{dt} = 8\pi r\frac{dr}{dt}$ ...(i)

It is given that $\frac{dS}{dt}$ = Cos t = k (say). Putting $\frac{dS}{dt}$ = k in (i), we get

k = $8\pi r\frac{dr}{dt}$

$\Rightarrow 8\pi$r dt = k dt [By separating the variables]

Integrating both sides, we have,

$4\pi r^2$ = kt + C ...(ii)

We are given that at t = 0, r = 3. Putting t = 0 and r = 3 in (ii), we have,

36π = k(0) + C ...(iii)

Putting t = 2 and r = 5 in (ii), we have,

100π = 2k + C ...(iv)

Solving (iii) and (iv), we get, C = 36π and k = 32π

Substituting the values of C and k in (ii), we get,

$4\pi r^2 = 32\pi t + 36\pi \Rightarrow r^2 = 8t + 9 \Rightarrow r = \sqrt{8t+9}$

34. The given differential equation is,

$x\frac{dy}{dx}$ = y - x $\cos^2\left(\frac{y}{x}\right)$

$\Rightarrow \frac{dy}{dx} = \frac{y - x\cos^2\left(\frac{y}{x}\right)}{x}$

This is a homogeneous differential equation

Putting y = vx and $\frac{dy}{dx} = v + x\frac{dv}{dx}$, we get

$v + x\frac{dv}{dx} = \frac{vx - x\cos^2 v}{x}$

$\Rightarrow v + x\frac{dv}{dx}$ = v - $\cos^2$ v

$\Rightarrow x\frac{dv}{dx}$ = $-\cos^2 v$

$\Rightarrow$ $\sec^2$ v dv = $-\frac{1}{x}$ dx

Integrating both sides, we get

$\int \sec^2$ v dv = $-\int \frac{1}{x}$ dx

$\Rightarrow$ tan v = - log |x| + log C

$\Rightarrow$ tan v = log $\left|\frac{C}{x}\right|$

Putting v = $\frac{y}{x}$, we get

$\therefore \tan\left(\frac{y}{x}\right) = \log\left|\frac{C}{x}\right|$

Hence, $\tan\left(\frac{y}{x}\right) = \log\left|\frac{C}{x}\right|$ is the required solution.

OR

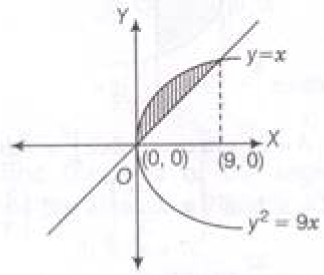
We have, y^2 = 9x and y = x

$\Rightarrow x^2 = 9x$

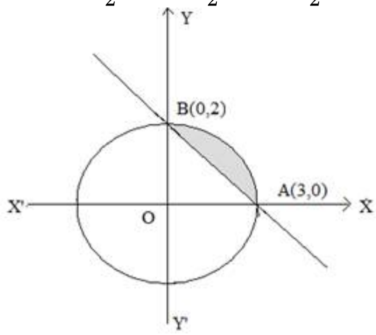
$\Rightarrow x^2 - 9x = 0$

$\Rightarrow x(x-9) = 0$

$\Rightarrow x = 0, 9$



$\therefore$ Area of shaded region, $A=\int_0^9\left(\sqrt{9x}-x\right)dx=\int_0^9 3x^{1/2}dx-\int_0^9 xdx$
$=\left(\frac{6x^{3/.2}}{3}-\frac{x^2}{2}\right)_0^9$
$=\left(6.\frac{9^{3/.2}}{3}-\frac{81}{2}-0\right)$
$=54-\frac{81}{2}=\frac{108-81}{2}=\frac{27}{2}$ sq units.

35. 

$\frac{x^2}{9}+\frac{y^2}{4}=1$
$\frac{x}{3}+\frac{y}{2}=1$
$\Rightarrow\ \frac{x^2}{(3)^2}+\frac{y^2}{(2)^2}=1$ is the equation of ellipse and
$\frac{x}{3}+\frac{y}{2}=1$ is the equation of intercept form of line
Area $=\frac{2}{3}\int_0^3\sqrt{9-x^2}dx-\frac{2}{3}\int_0^3(3-x)dx$
$=\frac{2}{3}\left[\frac{x}{2}\sqrt{9-x^2}+\frac{9}{2}sin^{-1}\frac{x}{3}-3x+\frac{x^2}{3}\right]_0^3$
$=\frac{2}{3}\left[(0+\frac{9}{2}(sin^{-1}(-1)-3(3)+\frac{9}{2})-0\right]$
$=\frac{2}{3}[\frac{9\pi}{4}-\frac{9}{2}]$
$=\frac{3}{2}(\pi-2)$sq. unit

Section III

36. To solve this we Use integration by parts that is,
$\int I\times II\,dx=I\int II\,dx-\int\frac{d}{dx}I\left(\int II\,dx\right)dx$
$y=x\int\cot x\,dx-\int\frac{d}{dx}x\left(\int\cot x\,dx\right)dx$
$y=(x\log\sin x)_0^{\frac{\pi}{2}}-\int_0^{\frac{\pi}{2}}\log\sin xdx$
Let, $I=\int_0^{\frac{\pi}{2}}\log\sin xdx$ .... (i)
Use King theorem of definite integral
$\int_a^b f(x)dx=\int_a^b f(a+b-x)dx$
$I=\int_0^{\frac{\pi}{2}}\log\sin\left(\frac{\pi}{2}-x\right)dx$
$I=\int_0^{\frac{\pi}{2}}\log\cos xdx$ ..... (ii)
Adding eq. (i) and (ii) we get,
$2I=\int_0^{\frac{\pi}{2}}\log\sin xdx+\int_0^{\frac{\pi}{2}}\log\cos xdx$
$2I=\int_0^{\frac{\pi}{2}}\log\frac{2\sin x\cos x}{2}dx$
$2I=\int_0^{\frac{\pi}{2}}\log\sin 2x-\log 2dx$
Let, 2x = t
$\Rightarrow$ 2dx =dt

At x = 0, t = 0

At $x=\frac{\pi}{2}, t=\pi$

$2I=\frac{1}{2}\int_0^\pi \log\sin t dt-\frac{\pi}{2}\log 2$

$2I=\frac{2}{2}\int_0^{\frac{\pi}{2}} \log\sin x dx-\frac{\pi}{2}\log 2$

$2I=I-\frac{\pi}{2}\log 2$

$I=\int_0^{\frac{\pi}{2}} \log\sin x dx=-\frac{\pi}{2}\log 2$

$y=(x\log\sin x)_0^{\frac{\pi}{2}}-\int_0^{\frac{\pi}{2}} \log\sin x dx$

$y=\frac{\pi}{2}\log\sin\frac{\pi}{2}-\left(-\frac{\pi}{2}\log 2\right)$

$y=\frac{\pi}{2}\log 2$

Hence proved..

OR

Given $\int_0^{\frac{\pi}{2}} \frac{\cos^2 x}{\cos^2 x+4\sin^2 x}dx$

Let $I=\int_0^{\frac{\pi}{2}} \frac{\cos^2 x}{\cos^2 x+4\sin^2 x}dx$ ......(i)

$\Rightarrow \mathrm{I}=\int_0^{\frac{\pi}{2}} \frac{\cos^2 \mathrm{x}}{\cos^2 \mathrm{x}+4(1-\cos^2 \mathrm{x})}\mathrm{dx}$

$=\int_0^{\frac{\pi}{2}} \frac{\cos^2 x}{\cos^2 x+4(1)-(4\cos^2 x)}dx$

= $\int_0^{\frac{\pi}{2}} \frac{\cos^2 x}{4-3\cos^2 x}dx$

= $\int_0^{\frac{\pi}{2}} \frac{\frac{1}{3}\cdot 3\cos^2 x}{4-3\cos^2 x}dx$

= $-\frac{1}{3}\cdot\int_0^{\frac{\pi}{2}} \frac{-3\cos^2 x+4-4}{4-3\cos^2 x}dx$

= $-\frac{1}{3}\cdot\int_0^{\frac{\pi}{2}} \frac{4-3\cos^2 x}{4-3\cos^2 x}dx+\frac{1}{3}\cdot\int_0^{\frac{\pi}{2}} \frac{4}{4-3\cos^2 x}dx$

= $-\frac{1}{3}\cdot\int_0^{\frac{\pi}{2}}(1)dx+\frac{1}{3}\cdot\int_0^{\frac{\pi}{2}} \frac{4}{4-3\left(\frac{1}{\sec^2 x}\right)}dx$

= $-\frac{1}{3}\cdot[\mathrm{x}]_0^{\frac{\pi}{2}}+\frac{1}{3}\cdot\int_0^{\frac{\pi}{2}} \frac{4\sec^2 \mathrm{x}}{4\sec^2 \mathrm{x}-3}\mathrm{dx}$

= $-\frac{1}{3}\cdot\left[\frac{\pi}{2}\right]+\frac{1}{3}\cdot\int_0^{\frac{\pi}{2}} \frac{4\sec^2 x}{4(1+\tan^2 x)-3}dx$

$\Rightarrow \mathrm{I}=-\frac{\pi}{6}+\frac{2}{3}\cdot\int_0^{\frac{\pi}{2}} \frac{2\sec^2 \mathrm{x}}{1+4\tan^2 \mathrm{x}}\mathrm{dx}$

$\Rightarrow \mathrm{I}=-\frac{\pi}{6}+\mathrm{I}_1$ .....(ii)

First solve for I_1:

$I_1=\frac{2}{3}\cdot\int_0^{\frac{\pi}{2}} \frac{2\sec^2 x}{1+4\tan^2 x}dx$

Let 2 tan x = t $\Rightarrow$ 2 $\sec^2$x dx dt

When x = 0 then t = 0 and when x = $\frac{\pi}{2}$ then t = ∞

$\Rightarrow \frac{2}{3}\cdot\int_0^{\frac{\pi}{2}} \frac{2\sec^2 x}{1+4\tan^2 x}dx=\frac{2}{3}\cdot\int_0^\infty \frac{1}{1+t^2}dt$

$\Rightarrow \mathrm{I}_1=\frac{2}{3}\left[\tan^{-1}\mathrm{t}\right]_0^\infty$

= $\frac{2}{3}\left[\tan^{-1}\infty-\tan^{-1}0\right]$

$\Rightarrow \mathrm{I}_1=\frac{2}{3}\cdot\frac{\pi}{2}$

$\Rightarrow \mathrm{I}_1=\frac{\pi}{3}$

Put this value in equ.(ii)

$\Rightarrow \mathrm{I}=-\frac{\pi}{6}+\frac{\pi}{3}$

$\Rightarrow \mathrm{I}=\frac{\pi}{6}$

37. According to the question ,

Given differential equation is

$ydx+x\log\left|\frac{y}{x}\right|dy-2xdy=0$

$\Rightarrow \quad ydx=\left[2x-x\log\left|\frac{y}{x}\right|\right]dy$

$\Rightarrow \quad \frac{dy}{dx} = \frac{y}{2x - x\log\left|\frac{y}{x}\right|}$

Now, let $F(x,y) = \frac{y}{2x - x\log\left|\frac{y}{x}\right|}$

On replace x by λx and y by λy both sides, we get

$F(\lambda x, \lambda y) = \frac{\lambda y}{2\lambda x - \lambda x\log\left|\frac{\lambda y}{\lambda x}\right|}$

$= \frac{\lambda y}{\lambda\left[2x - x\log\left|\frac{y}{x}\right|\right]}$

$\Rightarrow F(\lambda x, \lambda y) = \lambda^0 \frac{y}{2x - x\log\left|\frac{y}{x}\right|} = \lambda^0 F(x,y)$

So, the given differential equation is homogeneous.

On putting y = vx $\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$ in Eq. (i).

we get $v + x\frac{dv}{dx} = \frac{vx}{2x - x\log\left|\frac{vx}{x}\right|} = \frac{v}{2 - \log|v|}$

$\Rightarrow \quad x\frac{dv}{dx} = \frac{v}{2 - \log|v|} - v = \frac{v - 2v + v\log|v|}{2 - \log|v|}$

$\Rightarrow \quad x\frac{dv}{dx} = \frac{-v + v\log|v|}{2 - \log|v|}$

$\Rightarrow \quad \frac{2 - \log|v|}{v\log|v| - v}dv = \frac{dx}{x}$

On integrating both sides, we get

$\int \frac{2 - \log|v|}{v(\log|y| - 1)}dv = \int \frac{dx}{x}$

On putting log |v| = t $\Rightarrow \frac{1}{v}dv = dt$

Then, $\int \frac{2-t}{t-1}dt = \log|x| + C$

$\Rightarrow \int \left(\frac{1}{t-1} - 1\right) dt = \log|x| + C$

$\Rightarrow \quad \log|t - 1| - t = \log|x| + C$

$\Rightarrow \quad \log|\log v - 1| - \log v = \log|x| + C$ [put t = log |v|]

$\Rightarrow \quad \log\left|\frac{\log v - 1}{v}\right| = \log|x| + C \left[\because \log m - \log n = \log\left(\frac{m}{n}\right)\right]$

$\Rightarrow \log\left|\frac{\log v - 1}{v}\right| - \log|x| = C \Rightarrow \log\left|\frac{\log v - 1}{vx}\right| = C$

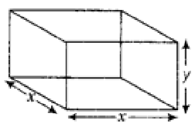
$\therefore \quad \log\left|\frac{\log\frac{y}{x} - 1}{y}\right| = c \left[\because y = vx \Rightarrow v = \frac{y}{x}\right]$

which is the required solution.

OR

Since, volume of the box = 1024 cm^3

Let length of the side of square base be x cm and height of the box be y cm.



$\therefore$ Volume of the box (V) $= x^2.y = 1024$

Since, $x^2y = 1024 \Rightarrow y = \frac{1024}{x^2}$

Let C denotes the cost of the box.

$\therefore C = 2x^2 \times 5 + 4xy \times 2.50$

$= 10x^2 + 10xy = 10x(x + y)$

$= 10x\left(x + \frac{1024}{x^2}\right)$

$= \frac{10x}{x^2}\left(x^3 + 1024\right)$

$\Rightarrow C = 10x^2 + \frac{1024}{x}$ ...(i)

On differentiating both sides w.r.t. x, we get

$\frac{dC}{dx} = 20x - 10240(x)^{-2}$

$= 20x - \frac{10240}{x^2}$ ...(ii)

Now, $\frac{dC}{dx} = 0$
$\Rightarrow 20x = \frac{10240}{x^2}$
$\Rightarrow 20x^3 = 10240$
$\Rightarrow x^3 = 512 = 8^3 \Rightarrow x = 8$
Again, differentiating Eq. (ii) w.r.t. x, we get
$\frac{d^2C}{dx^2} = 20 - 10240\,(-2)\,.\,\frac{1}{x^3}$
$= 20 + \frac{20480}{x^3}$
$\therefore \left(\frac{d^2C}{dx^2}\right)_{x=8} = 20 + \frac{20480}{512} = 60 > 0$
For x = 8, cost is minimum and the corresponding least cost of the box
$C\,(8) = 10.8^2 + \frac{10240}{8}$
$\therefore$ Least cost = ₹ 1920

38. According to the question ,
Given differential equation is
$\frac{dy}{dx} = \frac{y^2}{xy - x^2}$ ...(i)
Let $F(x, y) = \frac{y^2}{xy - x^2}$
Now, on replacing x by λx and y by λy , we get
$F(\lambda x, \lambda y) = \frac{\lambda^2 y^2}{\lambda^2 (xy - x^2)} = \lambda^0 \frac{y^2}{xy - x^2} = \lambda^0 F(x, y)$
Thus, the given differential equation is a homogeneous differential equation.
Now, to solve it, put y = vx
$\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$
From Eq. (i), we get
$v + x\frac{dv}{dx} = \frac{v^2 x^2}{vx^2 - x^2} = \frac{v^2}{v-1}$
$\Rightarrow \quad x\frac{dv}{dx} = \frac{v^2}{v-1} - v = \frac{v^2 - v^2 + v}{v-1}$
$\Rightarrow \quad x\frac{dv}{dx} = \frac{v}{v-1} \Rightarrow \frac{v-1}{v} dv = \frac{dx}{x}$
On integrating both sides, we get
$\int \left(1 - \frac{1}{v}\right) dv = \int \frac{dx}{x}$
$\Rightarrow \quad v - \log|v| = \log|x| + C$
$\Rightarrow \quad \frac{y}{x} - \log\left|\frac{y}{x}\right| = \log|x| + C \left[\text{ put } v = \frac{y}{x}\right]$
$\Rightarrow \quad \frac{y}{x} - \log|y| + \log|x| = \log|x| + C \left[\because \log\left(\frac{m}{n}\right) = \log m - \log n\right]$
$\therefore \quad \frac{y}{x} - \log|y| = C$
which is the required solution.

OR

Let P be the principal (amount) at the end of t years.
According to the given condition, rate of increase of principal per year = 5% (of principal)
$\Rightarrow \frac{dP}{dt} = \frac{5}{100} \times P$
$\Rightarrow \frac{dP}{dt} = \frac{P}{20}$
$\Rightarrow \frac{dP}{P} = \frac{dt}{20}$ [Separating variables]
Integrating both sides,
log P = $\frac{1}{20}$ t + c ...(i)
[Since P being principal > 0, hence $\log|P| = \log P$]
Now initial principal = ₹ 1000 (given), i.e., when t = 0 then P = 1000
Therefore, putting t = 0, P = 1000 in eq. (i), log 1000 = c
Putting $\log 1000 = c$ in eq. (i), log P = $\frac{1}{20}$ t + log 1000
$\Rightarrow$ log P - log 1000 = $\frac{1}{20}$t
$\Rightarrow$log $\frac{P}{1000} = \frac{1}{20}t$ ...(ii)
Now putting t = 10 years (given)

$\log \frac{P}{100} = \frac{1}{20} \times 10 = \frac{1}{2} = 0.5$

$\Rightarrow \frac{P}{1000} = e^{0.5}$ [∵ If x = t, then x = e^t

P = 1000 × 1.648 = ₹ 1648